

Dedicated to Bob Griess on the occasion of his 71st birthday.

GENERALIZED TWISTED QUANTUM DOUBLES OF A FINITE GROUP AND RATIONAL ORBIFOLDS

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ABSTRACT. In previous work the authors introduced a new class of modular quasi-Hopf algebra $D^\omega(G, A)$, associated to a finite group G , a central subgroup A and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. In the present paper we propose a description of the class of orbifold models of rational vertex operator algebras whose module category is tensor equivalent to $D^\omega(G, A)\text{-mod}$. The paper includes background on quasi-Hopf algebras and a discussion of some relevant orbifolds.

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1. INTRODUCTION

Since its introduction by Dijkgraaf, Pasquier and Roche [DPR], the *twisted quantum double* has been a source of inspiration in the related areas of *quasiHopf algebras*, *modular tensor categories* (MTC), and *orbifold models* of holomorphic vertex operator algebras (VOA). It follows from the work [Mu1] of Müger that the module category $D^\omega(G)\text{-mod}$ of a twisted quantum double of a finite group G (notation and further details are provided below) is a MTC. Even before the idea of a MTC existed, DPR had more-or-less conjectured [DPR] that if V is a holomorphic VOA admitting G as a group of automorphisms then $V^G\text{-mod}$ is equivalent to $D^\omega(G)\text{-mod}$ for some 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. Kirillov considered the conjecture from the perspective of fusion categories [Ki]. His work would imply the conjecture if one can show that all the g -twisted modules of V form a G -graded fusion category. This condition amounts to the rationality of V^G . With recent advances, this conjecture is now known for a large family of groups G . The recent work [MC] of Miyamoto and Carnahan proves that V^G is rational for any solvable group G . A complete solution to the conjecture seems to be within reach.

On the other hand, much less known in the case of *rational orbifolds*, i.e., orbifolds V^G where V is a *rational VOA*, but not necessarily holomorphic. Already in [DPR], the authors asked for a description of the $c=1$ ADE orbifolds V^G where $V=L(sl_2, 1)$ is the affine algebra of level 1 associated to sl_2 (alternatively, the rank 1 lattice theory $V_{\sqrt{2}\mathbb{Z}}$ associated with the A_1 root lattice) and G a finite subgroup of $SO(3, \mathbb{R})$. Until recently there was no really satisfactory answer to this question. Fusion rules and S - and T -matrices for these theories have long been known (the icosahedral case proved to be particularly intractable) but a quasiHopf algebra replacing the twisted double was missing. More generally, there does not seem to be even a conjectural description of $V^G\text{-mod}$ in the literature for any reasonably substantial class of rational orbifolds beyond the holomorphic case.

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In our recent work [MN1] we introduced a *generalization* of the twisted quantum double, denoted by $D^\omega(G, A)$, which is a certain quasiHopf algebra quotient of $D^\omega(G)$ obtained from a central subgroup A of G . (The case $A=1$ reduces to $D^\omega(G)$.) We gave necessary and sufficient conditions that $D^\omega(G, A)$ -mod is a MTC. The purpose of the present paper is to present a conjectural description of those rational orbifolds V^G whose module category is equivalent to some $D^\omega(G, A)$ -mod, and to discuss a few examples. These include the ADE examples mentioned above, thus providing an answer to the question raised by DPR.

The paper is organized as follows. In Section 2 we present a general discussion of quasiHopf algebras, with emphasis on the construction and properties of $D^\omega(G, A)$. In Section 3 we state our conjecture relating $D^\omega(G, A)$ to certain rational orbifolds, and in Section 4 we consider some illustrative examples.

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2. QUASIHOPF ALGEBRAS AND $D^\omega(G, A)$

In this Section we provide some background, taken from [MN1], about the quasiHopf algebras $D^\omega(G, A)$, which we call *generalized twisted quantum doubles*.

We use the following notation for a finite group G . $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times)$ is the group of *characters* of G , $x^g = g^{-1}xg$ ($x, g \in G$) is right conjugation in G , the *centralizer* of x in G is $C_G(x) := \{g \in G \mid x^g = x\}$, $Z(G) := \cap_x C_G(x)$ is the *center* of G .

We take our base field to be the complex numbers \mathbb{C} . A *quasiHopf algebra* is a tuple $(H, \Delta, \epsilon, \phi, \alpha, \beta, S)$, where H is a unital algebra and $\Delta: H \rightarrow H \otimes H$ an algebra morphism that is *quasicoassociative* in the sense that there is a map ϕ (the *Drinfeld associator*) making the following diagram commutative:

$$(2.1) \quad \begin{array}{ccc} & H \otimes H & \\ \Delta \otimes Id \swarrow & & \searrow Id \otimes \Delta \\ (H \otimes H) \otimes H & \xrightarrow{\phi} & H \otimes (H \otimes H) \end{array}$$

S is the *antipode*, ϵ the *counit*, and $\alpha, \beta \in H$ are certain distinguished elements. We generally suppress all reference to these elements of a quasiHopf algebra; this will not impair the reader's understanding of what follows. One also requires ϕ to satisfy some *cocycle* conditions in the form of certain diagrams involving fourfold tensor products of H that are required to commute. Again we will generally suppress such details. A Hopf algebra is a quasiHopf algebra for which $\alpha = \beta = 1$ and $\phi = 1 \otimes 1 \otimes 1$. For further background on quasiHopf algebras, see [D], [K] and [MN3].

One of the great virtues of a quasiHopf algebra H is that the category H -mod of (finite-dimensional) H -modules is a finite tensor category, though not necessarily a MTC.

Fix a finite group G . The group algebra $\mathbb{C}G$ is a familiar example of a Hopf algebra, the coproduct being defined by $\Delta(g) = g \otimes g$ ($g \in G$). Dualizing, the dual group algebra \mathbb{C}^G has basis e_g ($g \in G$) dual to the basis of group elements in G . It

is a Hopf algebra with product and coproduct defined by

$$e_g e_h = \delta_{g,h} e_g, \quad \Delta: e_g \mapsto \sum_{ab=g} e_a \otimes e_b.$$

Now fix a normalized multiplicative 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$. A basic example of a quasiHopf algebra is the *twisted dual group algebra* \mathbb{C}_ω^G , obtained from \mathbb{C}^G just by replacing $1 \otimes 1 \otimes 1$ by a more interesting Drinfeld associator defined by multiplication by

$$(2.2) \quad \phi := \sum_{a,b,c} \omega(a,b,c)^{-1} e_a \otimes e_b \otimes e_c.$$

Here, the cocycle conditions amount to the 3-cocycle identity satisfied by ω .

The *twisted quantum double* $D^\omega(G) = \mathbb{C}_\omega^G \otimes \mathbb{C}G$ occurs as the middle term of a sequence of morphisms of quasiHopf algebras

$$\mathbb{C}_\omega^G \xrightarrow{i} D^\omega(G) \xrightarrow{p} \mathbb{C}G$$

where $i(e_g) = e_g \otimes 1$, $p(e_g \otimes x) = \delta_{g,1} x$. $D^\omega(G)$ is itself a quasiHopf algebra with the following product and coproduct:

$$(e_g \otimes x) \cdot (e_h \otimes y) = \theta_g(x, y) \delta_{g^x, h} e_g \otimes xy$$

$$\Delta: e_g \otimes x \mapsto \sum_{ab=g} \gamma_x(a, b) e_a \otimes x \otimes e_b \otimes x.$$

The scalars $\theta_g(x, y), \gamma_x(a, b)$ are determined by ω as follows:

$$\theta_g(x, y) := \frac{\omega(g, x, y) \omega(x, y, g^{xy})}{\omega(x, g^x, y)},$$

$$\gamma_x(a, b) := \frac{\omega(a, b, x) \omega(x, a^x, b^x)}{\omega(a, x, b^x)}.$$

We remark that the 2-cochains defined by the θ 's and γ 's have subtle properties which govern much of the behaviour of $D^\omega(G)$ – for example, the fact that $D^\omega(G)$ really is a quasiHopf algebra, which is not obvious. As another example, if we restrict x, y to $C_G(g)$ then the 2-cochains θ_g, γ_g *coincide* and become 2-cocycles. We informally record this as

$$(2.3) \quad \gamma_g = \theta_g \in Z^2(C_G(g), \mathbb{C}^\times),$$

where it is understood that the domains of γ_g, θ_g are here restricted to $C_G(g)$.

Since $D^\omega(G)$ -mod is the Drinfeld center of $\text{Vec}(G, \omega)$, the fusion category of G -graded vector space with the associativity given by ω , it follows from [Mu1] that $D^\omega(G)$ -mod is a MTC.

We now fix another piece of data, namely a central subgroup $A \subseteq Z(G)$, and introduce

$$D^\omega(G, A) := \mathbb{C}_\omega^G \otimes \mathbb{C}(G/A).$$

Notice that because the conjugation action of A on G is *trivial* then the *identical* formulas used to define the operations in $D^\omega(G)$ still make sense in $D^\omega(G, A)$. When ω is *compatible* with A , one can equip $D^\omega(G, A)$ with a product and coproduct similar to the case of $D^\omega(G)$ so that $D^\omega(G, A)$ is a quasiHopf algebra.

Now it is natural to ask if, for suitable π' , there is a commuting diagram of quasiHopf algebras and morphisms

$$(2.4) \quad \begin{array}{ccccc} \mathbb{C}_\omega^G & \xrightarrow{i} & D^\omega(G) & \xrightarrow{p} & \mathbb{C}G \\ \downarrow = & & \downarrow \pi' & & \downarrow \pi \\ \mathbb{C}_\omega^G & \xrightarrow{i} & D^\omega(G, A) & \xrightarrow{\bar{p}} & \mathbb{C}(G/A) \end{array}$$

where $\pi(x)=xA$ and $\bar{p}(e_g \otimes xA)=\delta_{g,1}xA$ for $x \in G$. In case π' exists, it will satisfy $\pi'(e_g \otimes x)=\lambda e_g \otimes xA$ for a scalar λ that depends on g and x .

Generally there will be no such π' , but we can give necessary and sufficient conditions for its existence. To explain this we first consider the *group-like elements* of $D^\omega(G)$. These are the (nonzero) elements $u \in D^\omega(G)$ such that $\Delta(u)=u \otimes u$. As in the case of Hopf algebras, the group-like elements form a multiplicative subgroup $\Gamma^\omega(G) \subseteq D^\omega(G)^\times$ of the group of units. We are more interested in the *central group-like elements*, which is the subgroup $\Gamma_0^\omega(G) \subseteq \Gamma^\omega(G)$ consisting of elements that commute with all elements of $D^\omega(G)$. One can show (cf. [MN3]) that there is a diagram of short exact sequences of groups (actually central extensions)

$$\begin{array}{ccccccc} 1 & \longrightarrow & \widehat{G} & \xrightarrow{i} & \Gamma^\omega(G) & \xrightarrow{p} & B^\omega(G) \longrightarrow 1 \\ & & \uparrow = & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{i} & \Gamma_0^\omega(G) & \xrightarrow{p} & Z^\omega(G) \longrightarrow 1 \end{array}$$

where

$$B^\omega(G) := \{g \in G \mid \gamma_g \in B^2(G, \mathbb{C}^\times)\}, \quad Z^\omega(G) := B^\omega(G) \cap Z(G),$$

and vertical arrows are containments.

We emphasize that here, unlike the situation of (2.3), in order for g to lie in $B^\omega(G)$ it is necessary that the 2-cochain γ_g be a 2-coboundary on G rather than just the centralizer $C_G(g)$. On the other hand, if $g \in Z(G)$ then the context of (2.3) pertains, so that $\gamma_g = \theta_g$ is always a 2-cocycle on G , and the requirement to belong to $Z^\omega(G)$ is that this 2-cocycle is in fact a 2-coboundary.

It is shown in [MN1] that the existence of the morphism π' in (2.4) is equivalent to the existence of an enlarged diagram of central extensions

$$(2.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \widehat{G} & \xrightarrow{i} & \Gamma^\omega(G) & \xrightarrow{p} & B^\omega(G) \longrightarrow 1 \\ & & \uparrow = & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{i} & \Gamma_0^\omega(G) & \xrightarrow{p} & Z^\omega(G) \longrightarrow 1 \\ & & \uparrow = & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{i} & p^{-1}(A) & \xrightarrow{p} & A \longrightarrow 1 \end{array}$$

where vertical maps are again containments and the *lower ses splits*.

What is being asserted here is the following: the central subgroup $A \subseteq Z(G)$ is required to also lie in $B^\omega(G)$, i.e., the 2-cocycles θ_g for $g \in A$ are 2-coboundaries on G . Moreover, the ses obtained by pulling-back A along p must split.

Once π' is available, it follows that $D^\omega(G, A)$ -mod is a subcategory of $D^\omega(G)$ -mod. However, M\"uger's theorem will generally not hold for $D^\omega(G, A)$, that is to say $D^\omega(G, A)$ -mod is generally *not* a MTC. We will describe necessary and sufficient conditions, established in [MN1], that make this so.

First we say a bit more about the middle ses in (2.5). Given $g \in Z^\omega(G)$ we have $\gamma_g = \theta_g \in B^2(G, \mathbb{C}^\times)$, so that there is $\tau_g \in C^1(G, \mathbb{C}^\times)$ satisfying $\delta\tau_g = \theta_g$, i.e.,

$$\tau_g(x)\tau_g(y) = \theta_g(x, y)\tau_g(xy) \quad (x, y \in G).$$

Because θ_g is a 2-coboundary, the twisted group algebra $\mathbb{C}^{\theta_g}G$ that it defines is isomorphic to the group algebra $\mathbb{C}G$, and τ_g defines a choice of isomorphism

$$\mathbb{C}G \xrightarrow{\cong} \mathbb{C}^{\theta_g}G, \quad x \mapsto \tau_g(x)x.$$

There is no canonical choice of τ_g , but any two of them differ by a character $\chi \in \widehat{G}$. Indeed, we have

$$\Gamma_0^\omega(G) = \left\{ \sum_{x \in G} \tau_g(x) \chi(x) e_x \otimes g \mid \chi \in \widehat{G}, g \in Z^\omega(G) \right\}.$$

A 2-cocycle $\beta \in Z^2(Z^\omega(G), \widehat{G})$ that defines the central extension that is the middle ses in (2.5) is given by the formula

$$\beta(g, h)(k) = \frac{\tau_g(k)\tau_h(k)}{\tau_{gh}(k)} \theta_k(g, h) \quad (g, h \in Z^\omega(G), k \in G).$$

Because we are assuming that the lower ses in (2.5) splits, the restriction of β to A is a 2-coboundary on A . That is, there is a 1-cochain $\nu \in C^1(A, \widehat{G})$ such that

$$\beta(a, b) = \frac{\nu(a)\nu(b)}{\nu(ab)} \quad (a, b \in A).$$

Now we can show [MN1] that the formula

$$(2.6) \quad (a|b)_\nu := \frac{\tau_a(b)\tau_b(a)}{\nu(a)(b)\nu(b)(a)}$$

defines a *symmetric bicharacter* $(\mid)_\nu : A \times A \rightarrow \mathbb{C}^\times$. Using results of M\"uger [Mu1], [Mu2], it follows that $D^\omega(G, A)$ -mod is a MTC if, and only if, $(\mid)_\nu$ is *nondegenerate*. Actually, $(\mid)_\nu$ is the restriction of a natural bicharacter defined on $\Gamma_0^\omega(G)$, however we will not go into this here.

3. SIMPLE CURRENT ORBIFOLDS

In the spirit of the proposal in [DPR] that the module categories of holomorphic orbifolds coincide with the module categories $D^\omega(G)$ -mod, in this Section we describe those rational orbifolds V^G expected to have the property that V^G -mod is equivalent to some $D^\omega(G, A)$. In this setting, the case $A=1$ will reduce to the holomorphic orbifold case of DPR.

We will generally be lapse about the detailed properties of the VOAs which we consider, but we will be concerned with *rational* VOAs V , which at the very least means that V is a *simple* VOA and V -mod is a fusion category, i.e. a semisimple rigid tensor category with only finitely many simple objects. (See [DLM1] for further

background.) In fact we will only need a small subset of such theories, and to explain which ones these are we will review the theory of *simple currents*.

A simple current is a simple (or irreducible) V -module, call it M , with the property that for all simple V -modules N , the tensor product $M \boxtimes N$ is again a simple V -module. Otherwise stated, M represents an element in the Grothendieck group of V -mod, and an object of Frobenius-Perron dimension one. Thanks to the associativity of \boxtimes , the distinct (isomorphism classes of) simple currents form a group with respect to tensor product of modules, called the *group of simple currents*. It is an abelian group because V -mod is also braided. The identity element is, of course, the vacuum space V . We are concerned here with rational VOAs V with the property that *every simple V -module is a simple current* i.e., that V -mod is pointed. We call such a V a *simple current VOA*. There are many examples of such theories. In addition to holomorphic VOAs, where V is the *only* simple module, it is well-known that *lattice theories* V_L associated to a positive-definite even lattice L are also simple current VOAs. In this case, the group of simple currents is isomorphic to L^*/L where L^* is the *dual lattice* of L [D].

Given a simple current VOA V , we may form the sum of *all* simple V -modules M to obtain a larger space

$$\tilde{V} := \oplus_M M$$

\tilde{V} can be equipped with the structure of an *abelian intertwining algebra* in the sense of Dong-Leowsky [DL]. See [DLM2] for further details.

Conjecture. Suppose that V is a simple current VOA with group of simple currents A . Let $F \subseteq \text{Aut}(V)$ be a finite group of automorphisms of V (the *orbifold group*) such that F leaves invariant every simple V -module. Then there is *central extension*

$$1 \longrightarrow A \longrightarrow G \longrightarrow F \longrightarrow 1$$

and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ such that V^F -mod is equivalent to $D^\omega(G, A)$ -mod as modular tensor categories. Conversely, if $D^\omega(G, A)$ exists and $D^\omega(G, A)$ -mod is a MTC, then there is a simple current VOA V with group of simple currents A and a group of automorphisms $F=G/A$ such that V^F -mod is tensor equivalent to $D^\omega(G, A)$ -mod.

4. EXAMPLES

We illustrate the Conjecture by discussing some examples in greater detail. Let the notation be as before.

Example 1. The holomorphic case.

Here, $A=1$ means that V is a holomorphic VOA and $D^\omega(G, A)=D^\omega(G)$. The Conjecture thus reduces that of DPR.

Example 2. The case $F=1$, i.e., $G=A$ and $D^\omega(G, A)=\mathbb{C}_\omega^G$ as quasiHopf algebras.

We saw before that in order for π' to exist (i.e., $D^\omega(G, A)$ is a quasiHopf algebra quotient of $D^\omega(G)$) it is necessary that $A \subseteq Z^\omega(G)$, meaning that each $\theta_g \in B^2(G, \mathbb{C}^\times)$ is a 2-coboundary for all $g \in A$. That is, $\omega \in Z^3(A, \mathbb{C}^\times)_{ab}$ is an *abelian* 3-cocycle on A in the sense of [MN3], where such cocycles are studied extensively. They are closely related to the abelian cohomology groups introduced by Eilenberg and MacLane [EM].

Thus in the case at hand, the Conjecture asserts that $V\text{-mod}$ is tensor equivalent to \mathbb{C}_ω^A for an abelian 3-cocycle ω that is nondegenerate in a suitable sense. Rather than explain what degeneracy means here, we consider a special case in more detail.

Let L be an even lattice with bilinear form $\langle \cdot, \cdot \rangle: L \times L \rightarrow \mathbb{Z}$, and let V_L be the corresponding lattice VOA. Then $A = L^*/L$. The conjecture says that $V_L\text{-mod}$ is tensor equivalent to the dual group algebra of L^*/L twisted by an abelian 3-cocycle $\omega \in Z^3(L^*/L, \mathbb{C}^\times)$. The origin of ω is well-known in this case (cf. [DL], Chapter 12 and [MN3], Section 11). Let $s: A \rightarrow L^*$ be a normalized section of the canonical ses

$$1 \longrightarrow L \longrightarrow L^* \longrightarrow A \longrightarrow 1$$

Let $c_0: L^* \times L^* \rightarrow \mathbb{C}^\times$ be an *alternating bicharacter* with $c_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ ($\alpha, \beta \in L$). Set

$$\omega(a, b, c) = (-1)^{\langle s(c), s(a) + s(b) - s(a+b) \rangle} c_0(s(c), s(a) + s(b) - s(a+b)).$$

ω is indeed an abelian 3-cocycle (cf. [MN3], Proposition 11.1) whose cohomology class is *independent of the choice of bicharacter and section*.

Example 3. The case $|A|=2$.

Here, we are discussing simple VOAs V with just two simple modules. One of them is the adjoint module V , the other we denote by M . Roughly speaking, we may think of $\tilde{V} = V \oplus M$ as a holomorphic *super* VOA, though this may not conform to some definitions. (This will not matter in the following discussion.)

Next we discuss results of [MN4] concerning the existence of $D^\omega(G, A)$ in the special case that G is a finite group with *exactly one* subgroup of order 2. We take A to be the unique subgroup of order 2. Let T be a 2-Sylow subgroup of G . It is well-known that T is either cyclic or generalized quaternion. Groups with a unique involution have 2-periodic cohomology by the Artin-Tate theory [CE], and for such groups the 2-Sylow subgroup of $H^3(G, \mathbb{C}^\times)$ is *cyclic* of order $|T|$. We call a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ a *2-generator* if the corresponding cohomology class $[\omega] \in H^3(G, \mathbb{C}^\times)$ has order *divisible by* $|T|$.

In the setting of the previous paragraph, we can prove [MN4] that π' always exists, so that $D^\omega(G, A)$ is a quasiHopf algebra quotient of $D^\omega(G)$. Moreover, $D^\omega(G, A)\text{-mod}$ is a MTC if, and only if, ω is a 2-generator.

There are a number of interesting classes of finite groups with a unique subgroup A of order 2. The following lists some of them.

$$(4.7) \quad \begin{aligned} & SL_2(q) \quad (q \geq 3 \text{ an odd prime power}) \\ & 2.A_6, 2.A_7, 6.A_6, 6.A_7 \\ & \text{binary polyhedral groups} = \text{finite subgroups of } SU_2(\mathbb{C}) \end{aligned}$$

Here, $SL_2(q)$ is the group of 2×2 matrices of determinant 1 over the finite field of cardinality q , and $2.A_n$ is the 2-fold perfect central extension of the alternating group A_n . For $n=6, 7$ Schur discovered that there are exceptional 6-fold perfect central extensions of A_n . There are a few overlaps among these groups: $SL_2(3)$ and $SL_2(5)$ are the binary octahedral and icosahedral groups respectively, and $2.A_6 \cong SL_2(9)$.

Our Conjecture says that for each pair (G, ω) such that G has a unique subgroup of order 2 and $[\omega]$ is a 2-generator of $H^3(G, \mathbb{C}^\times)$, there is a holomorphic super VOA

$\tilde{V}=V \oplus M$ (in the sense described before) such that $G \subseteq \text{Aut}(\tilde{V})$, $G/A \subseteq \text{Aut}(V)$ and $V^{G/A}\text{-mod}$ is tensor equivalent to $D^\omega(G, A)\text{-mod}$.

As far as we know, the existence of a \tilde{V} for most groups G on the list (4.7) is unknown. However, it is well-known that the binary polyhedral groups, and indeed $SU_2(\mathbb{C})$ itself, act on the $c=1$ holomorphic super VOA defined by the A_1 root lattice theory $V_{\sqrt{2}\mathbb{Z}}$, also known as the affine algebra $L_{sl_2,1}$ of level 1 associated to the Lie algebra $sl_2(\mathbb{C})$. We review some of the details.

Adopting the lattice perspective, the irreducible modules for $V_{\sqrt{2}\mathbb{Z}}$ consist of the adjoint module $V_{\sqrt{2}\mathbb{Z}}$ and the module $M=V_{1/\sqrt{2}+\sqrt{2}\mathbb{Z}}$ corresponding to the two cosets of $\sqrt{2}\mathbb{Z}$ in its dual lattice $\frac{1}{\sqrt{2}}\mathbb{Z}$. Thus $\tilde{V}=V_{\sqrt{2}\mathbb{Z}} \oplus V_{1/\sqrt{2}+\sqrt{2}\mathbb{Z}}=V_{\frac{1}{\sqrt{2}}\mathbb{Z}}$. We have $\text{Aut}(V)=SO_3(\mathbb{R})$, obtained by exponentiating the weight 1 states of V (which form the Lie algebra sl_2). The projective action of this group on M lifts to a linear action of its universal cover $SU_2(\mathbb{C})$, which is the full automorphism group of \tilde{V} .

The binary polyhedral groups G of even order are the finite subgroups of $SU_2(\mathbb{C})$ that contain the center $\{\pm 1\}$. Setting $A=\{\pm 1\}$, the preceding discussion shows that $G \subseteq \text{Aut}(\tilde{V})$ and $G/A \subseteq \text{Aut}(V)$, so we have rational orbifolds $V^{G/A}$. These theories have been studied extensively in both the physical and mathematical literature, and the S - and T -matrices are known. See, for example, [CG] and [DV] for type D . Dong and Nagatomo [DN] computed the fusion rules for $V^{G/A}$ on the basis of the VOA axioms in all cases except the icosahedral example.

Now we can compute the S - and T -matrices for all $D^\omega(G, A)$ whenever G is a binary polyhedral group and ω is a 2-generator. Note that $H^3(G, \mathbb{C}^\times)$ is cyclic of order $|G|$ in this case, so that the number of 2-generators is exactly $|G|/2$. Consider, for example the icosahedral group $G=SL_2(5)$. This is a group of order 120, so there are 60 2-generators $[\omega]$ of order 8, 24, 40 or 120. In the case of $V^{G/A}\text{-mod}$ discussed above, the S -matrix appears to correspond to an $[\omega]$ of maximal order 120.

In this way we get a large number of MTCs and accompanying modular data, and the Conjecture says that among them we should find the categories $V^{G/A}\text{-mod}$. This appears to indeed be the case, since in particular we can find matching S -matrices in all cases.

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